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MONOMIAL CONDITIONS ON PRIME RINGS

BY

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ABSTRACT

The study of pivotal monomials (and related conditions) is continued and extended, with the aim of studying carefully a situation generalizing Martindale's theory of prime rings with generalized polynomial identity. This is used to describe various classes of rings in terms of simple elementary sentences. The focus is on prime "Johnson" rings, which play a crucial role in our characterizations. It turns out that these rings can be characterized in terms of generalized pivotal monomials, thereby yielding a theory similar to that of [17].

Introduction

A very useful tool in the structure theory of prime rings is Martindale's theorem [15], that the central closure of a prime ring with proper generalized identity is primitive with minimal nonzero left (and right) ideals. Let $\mathscr{C}_1$ = {prime rings with proper generalized identity}, $\mathscr{C}_2$ = {prime rings with a left ideal which is a *PI*-ring}, $\mathscr{C}_3$ = {prime rings with a right ideal which is a *PI*-ring}, and $\mathscr{C}_4$ = {rings whose maximal left quotient ring is an endomorphism ring of a left vector space over a central simple division ring}. Using Martindale's theorem, Jain [10] showed that $\mathscr{C}_1 = \mathscr{C}_2 = \mathscr{C}_3 = \mathscr{C}_4$ and proved that no ring in $\mathscr{C}_1$ has a nonzero nil left (or right) ideal.

It is natural to weaken $\mathscr{C}_4$ to {rings whose maximal left quotient ring is an endomorphism ring of a left vector space over a division ring}, and to ask whether Jain's results can be extended to this class of rings. Such extensions will be given in §§1 and 2. The method is to study prime Johnson rings "internally," easily obtaining some useful properties.

In §3, we will indicate how these results enable us to describe prime rings in terms of various elementary sentences (of type $\exists\forall\exists$). This continues the program of [2], [15], [16], [17] (and part of *PI*-theory), of describing the structure of rings in terms of simple elementary sentences.

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In order to be consistent with [17], we will be concerned with the category of rings with left module (R-mod), although many results quoted are actually proven in the symmetrical situation, in Mod-R, in the cited papers. Hopefully, no confusion should arise thereby.

§1. Essential subrings of primitive rings with socle

All rings are associative, not necessarily with 1, and "module" means left module. R will always be a ring. An R-module $M \supseteq R$ is an *essential extension* of R if $R \cap M' \neq 0$ for each nonzero submodule M' of M. A left ideal J of R is *large* if $J \cap B \neq 0$ for every nonzero left ideal B of R. For a subset V of an R-module M, define $Vx^{-1} = \{r \in R \mid rx \in V\}$, x in M. It is known (and easy to show) that if M is an essential extension of R, then Rx^{-1} is large, for all x in M.

For subsets S, V of R, define $\mathrm{Ann}_S V = \{r \in S \mid Vr = 0\}$, and $\mathrm{Ann}'_S V = \{r \in S \mid rV = 0\}$. If $a \in R$, we write $\mathrm{Ann}\, a$ and $\mathrm{Ann}'a$ for $\mathrm{Ann}_R\{a\}$ and $\mathrm{Ann}'_R\{a\}$; note that $\mathrm{Ann}'a = 0a^{-1}$. The left *singular ideal* $Z(R) = \{r \in R \mid \mathrm{Ann}'r \text{ is large}\}$, seen to be an ideal of R. Johnson has shown (cf. [12]) that, in the case $Z(R) = 0$, the injective hull $Q(R)$ of R can be provided with a ring structure which agrees with the R-module operations; motivated by this result, call R an *essential subring* of a ring $R' \supseteq R$ if R' is an essential extension of R, with respect to the induced R-module structure. (In this case, R' is sometimes called a left quotient ring of R, but this terminology is also used in other situations in the literature, and will not appear further in this paper.) A nonzero left ideal of R is *uniform* if it does not contain a direct sum of two nonzero left ideals of R.

THEOREM (Johnson [12, theor. 3.1]). *If R is a prime ring with uniform left ideal and if $Z(R) = 0$, then $Q(R)$, the maximal left quotient ring of R, is a ring of endomorphisms of a left vector space over a division ring.*

(Johnson actually proved a more general result, attributing the theorem quoted above to Lambek [13, theor. 4.2]; perhaps the results of this paper could be extended to Johnson's broader context of "irreducible" rings (cf. [12]).) A ring is *primitive* if it has a faithful irreducible left module. Also, the *socle* of a ring is 0 unless the ring has a minimal nonzero left ideal, in which case the socle is the sum of the minimal nonzero left ideals. Now, let $\mathrm{soc}(R)$ denote the socle of R. By [9, theor. 2, p. 65], if R is semiprime, then $\mathrm{soc}(R) = 0$ unless R has minimal nonzero right ideals, in which case $\mathrm{soc}(R)$ is the sum of the minimal nonzero *right* ideals of R; thus $\mathrm{soc}(R)$ is an ideal. Call R a (left) *Johnson ring* if

R is an essential subring of a ring of endomorphisms of a left vector space over a division ring. The Lambek–Johnson theorem then states, "Any prime ring with uniform left ideal and with zero left singular ideal is Johnson." (This definition of "Johnson" is more general than Faith's [6], which also requires the analogous condition on the right.)

Let R be prime with left ideal V and right ideal V'. Let $U = \mathrm{Ann}_V V$, $U' = \mathrm{Ann}'_{V'} V'$, $W = V/U$, $W' = V'/U'$. W and W' are easily seen to be prime rings.

PROPOSITION 1. (i) *If W or W' has uniform left ideal, then R has a uniform left ideal;*

(ii) *if $Z(R) \neq 0$, then $Z(W) \neq 0$ and $Z(W') \neq 0$.*

PROOF. (i) First suppose L/U is a uniform left ideal of W, and let $L' = VL$, a nonzero left ideal of R. We claim L' is uniform. Indeed, suppose $B_1, B_2 \subseteq L'$ are nonzero left ideals of R. Then $VB_1 \neq 0$ and $VB_2 \neq 0$, so $B_1, B_2 \not\subseteq U$. Since L/U is uniform, we have b_i in $B_i - U$, such that $(b_1 - b_2) \in U$. Hence $0 \neq Vb_1 \subseteq B_1 \cap B_2$, proving L' is uniform.

Next, suppose L/U' is a uniform left ideal of W'. Pick v in V' such that $Lv \neq 0$, and let $L' = RLv$. We claim that L' is uniform. Indeed, suppose $B_i \subseteq L'$ are left ideals of R, $i = 1, 2$, and let $B'_i = V'B_iv^{-1} \cap L$. Clearly $0 \neq V'B_i \subseteq V'RLv \subseteq Lv$, implying $(B'_i + U')/U'$ are nonzero left ideals of W'. Hence, there exist x_i in $B'_i - U'$, such that $(x_1 - x_2) \in U'$. Then $0 \neq x_1v \in V'B_1 \cap V'B_2 \subseteq B_1 \cap B_2$, proving L' is uniform.

(ii) Let $Z_1 = (Z(R)V + U)/U$. Clearly, $Z_1 \neq 0$, since R is prime. Moreover, for every $x \in Z(R)V$ and every nonzero left ideal B/U of V/U we have $VB \cap \mathrm{Ann}'x \neq 0$, so $(VB \cap \mathrm{Ann}'x)^2 \neq 0$, implying $(VB \cap \mathrm{Ann}'x) \not\subseteq U$. Thus, $B \cap Ux^{-1} \not\subseteq U$. This proves $(x + U) \in Z(W)$. Hence, $Z(W) \supseteq Z_1 \neq 0$.

Next, let $Z_2 = (V'Z(R) + U')/U' \neq 0$. Pick arbitrarily v in V' and z in $Z(R)$. For any nonzero left ideal B/U' of W', we claim that $B \cap \mathrm{Ann}'(vz) \not\subseteq U'$. This is obvious if $Bv = 0$, so assume $Bv \neq 0$. Then $V'(RBv \cap \mathrm{Ann}'z) \neq 0$; since $V'RB \subseteq B$, we have nonzero bv in Bv such that $bvz = 0$. This yields b in $(B \cap \mathrm{Ann}'(vz)) - U'$, proving the claim. It follows that $vz + U' \in Z(W')$. Therefore $Z(W') \supseteq Z_2 \neq 0$. Q.E.D.

COROLLARY 2. *If either W or W' has uniform left ideal and if either $Z(W) = 0$ or $Z(W') = 0$, then R is Johnson.*

A prime ring is *Goldie* if it is a (classical) left order in a simple artinian ring (cf. [7, 8]). Goldie [8] has characterized Goldie rings by the following two properties:

the left singular ideal is 0, and every nonzero left ideal contains a uniform left ideal. Hence, with the above notation, we have

COROLLARY 3. *If W or W' is Goldie, then R is Johnson.*

The remainder of this section will be spent in examining prime Johnson rings, obtaining converses of the above results and developing structure-theoretic tools. For the remainder of the discussion through Theorem 4, let R be a prime Johnson ring, i.e., R is an essential subring of a ring R' of endomorphisms of a left vector space over a division ring. Then R' is primitive and $\operatorname{soc}(R') \neq 0$. Let M be a faithful irreducible left R'-module, with $D = \operatorname{End}_{R'} M$. Note that $R' \subseteq \operatorname{End} M_D$, and $\operatorname{soc}(R') = \{\text{finite-ranked transformations of } \operatorname{End} M_D\}$, by [9, Structure Theorem, p. 75].

Let $\{y_i\}$ be a D-basis of M. For each i, define e_i in $\operatorname{soc}(R')$ by $e_i(\Sigma_j y_j d_j) = y_i d_i$. Also, for each d in D, define $d' \in \operatorname{End} M_D$ by $d'(\Sigma_j y_j d_j) = \Sigma_j y_j d d_j$. Clearly, the map $d \to d'$ is an isomorphism of D with a division subring D' of $\operatorname{End} M_D$; $[e_i, d'] = 0$ for all d' in D', all e_i. Although we need not have $d' \in R'$, it is obvious that for any r in $\operatorname{soc}(R')$, we have $rD' \subseteq r \operatorname{soc}(R')$ and $D'r \subseteq \operatorname{soc}(R')r$. We can find large left ideals J_i of R such that $0 \neq J_i e_i \subseteq R$; since R is prime, we can choose r_i in $J_i e_i$ such that $J_i e_i r_i \neq 0$. (Of course, $r_i \in \operatorname{soc} R'$.) Then $r_i y_i = \Sigma_j y_j d_{ji}$, suitable d_{ji} in D, with $d_{ii} \neq 0$; $r_i y_k = 0$ for $i \neq k$. Note that $r_i d' r_i = r_i d' d'_{ii}$ for all d' in D'. In particular, $r_i (d'_{ii})^{-1}$ and $(d'_{ii})^{-1} r_i$ are rank 1 idempotents, and $r_i^2 = r_i d'_{ii}$.

THEOREM 4. (i) *If $ar_i b \neq 0$, for suitable a, b in R', then $ar_i^t b \neq 0$ for all $t \geqq 1$;*

(ii) *$r_i R' r_i = r_i D' r_i$, which is a division ring;*

(iii) *$r_i R r_i$ is a (classical) left order in $r_i R' r_i$;*

(iv) *for $V = R' r_i$, $V_1 = R r_i$, $V' = r_i R'$, $V'_1 = r_i R$, the rings $V/\operatorname{Ann}_V V$ and $V'/\operatorname{Ann}_{V'} V'$ are division rings in which $V_1/\operatorname{Ann}_{V_1} V_1$ and $V'_1/\operatorname{Ann}_{V'_1} V'_1$ are left orders.*

PROOF. (i) Suppose $ar_i b \neq 0$. Then, for some y_k in M, $ar_i b y_k \neq 0$, so $0 \neq r_i b y_k = r_i e_i b y_k = r_i (y_i d)$ for suitable d in D. But then $ar_i^t b y_k (d^{-1} d_{ii}^{1-t} d) = ar_i^t y_i d_{ii}^{1-t} d = ar_i (d'_{ii})^{t-1} y_i d_{ii}^{1-t} d = ar_i y_i d = ar_i b y_k \neq 0$, so $ar_i^t b \neq 0$.

(ii) First we show that $r_i D' r_i$ is a division ring. For any d'_1, d'_2 in D', $(r_i d'_1 r_i)(r_i d'_2 r_i) = r_i d'_1 (d'_{ii})^2 d'_2 r_i \in r_i D' r_i$. Clearly $r_i (d'_{ii})^{-2} r_i$ is the identity, and $r_i d' r_i$ has left and right inverses $r_i (d'_{ii})^{-2} (d')^{-1} (d'_{ii})^{-2} r_i$, so $r_i D' r_i$ is indeed a division ring. Now choose r in R' and let $r r_i y_i = \Sigma_j y_j d_j$. Then $r_i r r_i y_i = r_i y_i d_i = r_i d'_i (d'_{ii})^{-1} r_i y_i$; since $r_i y_k = 0$ for each $k \neq i$, we conclude that $r_i r r_i = r_i d'_i (d'_{ii})^{-1} r_i \in r_i D' r_i$, implying $r_i R' r_i = r_i D' r_i$.

(iii) Suppose $r_i a r_i \neq 0$, a in R'. By (ii), there exists d' in D' such that

$(r_i d' r_i)(r_i^2) = r_i a r_i$. By (i), $r_i^2 d' r_i^2 \neq 0$, implying there is a large left ideal J of R with $0 \neq J r_i^2 d' r_i^2 \subseteq R$. Then $0 \neq (r_i J r_i)(r_i a r_i) \subseteq r_i R r_i$, proving that $r_i R r_i$ is a left order in $r_i R' r_i$.

(iv) For x in V, let $\bar{x}$ denote the image of x in $\bar{V} = V/\mathrm{Ann}_V V$. We can show that $\bar{V}$ is a division ring, by direct computation. Indeed, $\overline{(d'_{ii})^{-1} r_i}$ is the identity; also, for any nonzero $\overline{rr_i}$ in $\bar{V}$, we can find d' in D' such that $r_i r r_i = r_i d' r_i$, and then $\overline{(d'_{ii} d' d'_{ii})^{-1} r_i}$ is the left and right inverse of $\overline{rr_i}$. The other assertions are proved in similar manner, as consequences of (ii) and (iii). Q.E.D.

It should be noted that Faith and Chase have investigated essential subrings of the form $\mathrm{End}\, M_D$, for D a division ring and M a D-vector space, cf. [6, pp. 106–7]. Theorem 4 (ii) and (iii) follow easily from their results.

An *Ore domain* is a left order in a division ring. (Note that all Ore domains are Goldie.) Putting together our previous results yields immediately

THEOREM 5. *The following are equivalent for a prime ring R:*

(i) *R is Johnson;*

(ii) *R has a principal left ideal V such that $V/\mathrm{Ann}_V V$ is an Ore domain;*

(iii) *R has a principal right ideal V such that $V/\mathrm{Ann}'_V V$ is an Ore domain;*

(iv) *R has a left ideal V such that $V/\mathrm{Ann}_V V$ is Goldie;*

(v) *R has a right ideal V such that $V/\mathrm{Ann}'_V V$ is (left) Goldie;*

(vi) *$Z(R) = 0$ and R has a uniform left ideal.*

PROOF. We have (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), by Theorem 4; (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) are trivial; (iv) $\Rightarrow$ (vi) and (v) $\Rightarrow$ (vi), by Proposition 1. Finally, (vi) $\Rightarrow$ (i) by Johnson's theorem. Q.E.D.

Theorem 4 can be used to obtain even more information about prime Johnson rings. In the notation of Theorem 4, let $J_{i1} = \{x \in J_i e_i \mid e_i x \neq 0\}$ and $J_{i2} = \{x \in J_i e_i \mid e_i x = 0\}$. If $x_1 \in J_{i1}$ and $x_2 \in J_{i2}$, then clearly, $x_1 + x_2 \in J_{i1}$, i.e., $J_{i1} + J_{i2} \subseteq J_{i1}$.

LEMMA 6. *Suppose R is a prime Johnson ring with nonzero elements a, b. Then, in the above notation, there exists an element x in some J_{i1}, such that $axb \neq 0$.*

PROOF. Clearly $e_i b \neq 0$ for some i, implying $J_i e_i b \neq 0$ (since $Z(R) = 0$), so $aJ_i e_i b \neq 0$. Assume $aJ_{i1} b = 0$. Then $aJ_{i2} b \neq 0$, implying $0 \neq aJ_{i1} b + aJ_{i2} b = a(J_{i1} + J_{i2})b \subseteq aJ_{i1} b = 0$, a contradiction. Hence $aJ_{i1} b \neq 0$. Q.E.D.

Now $x \in J_{i1} \Leftrightarrow J_i e_i x \neq 0$ (since $Z(R) = 0$). Hence, we may replace r_i by x, in the notation of Theorem 4. This procedure yields

PROPOSITION 7. *If a, b are nonzero elements of a prime Johnson ring R, then there exists x in R such that* $ax^kb \neq 0$, *all* $k > 1$.

PROOF. Immediate consequence of Lemma 6 and Theorem 4(i).

PROPOSITION 8. *If R is a prime Johnson ring, then R has no nonzero nil left or right ideals. In fact, given a nonzero left or right ideal B of R, we can find x in some* J_{i1} *such that xBx is a nonzero domain.*

PROOF. Suppose B is a nonzero left ideal of R. By Lemma 6, there exists x in some J_{i1} such that $BxB \neq 0$. Hence $Bx \neq 0$, so $xBx \neq 0$. But xBx is a domain, by Theorem 4(ii); thus x^2B (and therefore B) is non-nil. The proof is symmetrical if B is instead a right ideal of R. Q.E.D.

Recall that $Z(R) = 0$ for any prime Johnson ring (cf. Theorem 5).

REMARK 9. Suppose R is an essential subring of a primitive ring R' with nonzero socle, and R' is dense in $\operatorname{End} M_D$. If $Z(R) = 0$, then R is an essential subring of $\operatorname{End} M_D$.

PROOF. Let $R'' = \operatorname{End} M_D$, and choose $r'' \in R''$. Then $0 \neq r'r'' \in \operatorname{soc} R'(= \operatorname{soc} R'')$ for some r' in $\operatorname{soc} R'$, so $0 \neq Jr'r'' \subseteq R$ for some large left ideal J of R. Thus, replacing r' by a suitable element of Jr', we have $0 \neq r'r'' \in R$. But $0 \neq J'r' \subseteq R$ for some large left ideal J' of R, and $J'r'r'' \subseteq R$. Hence $(Rr'' \cap R) \supseteq J'r'r'' \neq 0$ (since $Z(R) = 0$). This proves that R is an essential subring of R''. Q.E.D.

By this remark, it follows that the maximal left quotient ring of R (cf. [12]) is in fact $\operatorname{End} M_D$, so the ring of endomorphisms of a *left* vector space over a division ring in [12, theor. 3.1] is also a ring of endomorphisms of a *right* vector space over a division ring. We assume henceforth that $R' = \operatorname{End} M_D$.

Appendix to §1

Some of the above results generalize theorems of Jain [10], who based his proofs on Martindale's theory of central closure of a prime ring [15]. It may then be enlightening to link Martindale's central closure to the above theory. Suppose R is a prime Johnson ring, an essential subring of $R' = \operatorname{End} M_D$, notations as before. Let T be the subring of R' generated by R and D'. Clearly, R is an essential subring of T, so T is prime. Moreover, T has idempotents $r_i(d'_{ii})^{-1}$; one can prove easily that the Tr_i are minimal left ideals of T, so T is primitive with nonzero socle. Let $C = \operatorname{Cent}(D')$, and let RC be the subring of T generated by

R and C. R is essential subring of RC, so RC is prime; moreover, clearly $C \subseteq \mathrm{Cent}\, R'$. However, much more is true:

THEOREM 10. (i) *If* $\gamma \in R'$ *and* $[\gamma, R] = 0$, *then* $\gamma \in C$.

(ii) $C = \mathrm{Cent}\, RC = \mathrm{Cent}\, T = \mathrm{Cent}\, R'$.

(iii) *Suppose* $a \neq 0$ *and* b *are elements in* R' *such that* $arb = bra$ *for all* r *in an ideal* B *of* R. *Then* $b = \gamma a$ *for some* γ *in* C.

(iv) *There is an onto isomorphism* $\varphi: RC \to$ *central closure of* R, *such that* $\varphi(C) =$ *extended centroid of* R.

PROOF. (i) Given $u \in R'$, choose a large left ideal J of R such that $Ju \subseteq R$. Then, for any x in J, $xu\gamma = \gamma xu = x\gamma u$, so $J[u, \gamma] = 0$. Since $Z(R) = 0$, it follows easily that $[u, \gamma] = 0$; hence $\gamma \in \mathrm{Cent}\, R'$. By definition (of D), there exists d in D such that $\gamma y = yd$ for each y in M. In particular, $\gamma y_i = y_i d$, each i, so $\gamma = d'$. On the other hand, for every c in D, each y_i, $[\gamma, c']y_i = \gamma(y_i c) - c'(y_i d) = y_i cd - y_i cd = 0$. Therefore $\gamma \in C$.

(ii) Immediate consequence of (i).

(iii) Define $\gamma: M \to M$ by $\gamma(\Sigma_j r'_j a y_j d_j) = \Sigma_j r'_j b y_j d_j$, all r'_j in R', all d_j in D. Clearly $\mathrm{dom}(\gamma) = M$, since M is R'-irreducible, but we still must show that γ is well-defined. Well, suppose $\Sigma_{j=1}^{v} r'_j b y_j d_j \neq 0$. Then $e_k\, \Sigma_{j=1}^{v} r'_j b y_j d_j \neq 0$ for some k. One can find a large left ideal J of R such that $Je_k r'_j \subseteq R$, $1 \leqq j \leqq v$, and $Je_k r'_i \neq 0$ for some i. Now $0 \neq J'a \subseteq R$ for some J'. $J'aBJe_k r'_i \neq 0$, so $aBJe_k \neq 0$. Thus, $aBJe_k\, \Sigma_{j=1}^{v} r'_j b y_j d_j \neq 0$, implying, by hypothesis, $bBJe_k\, \Sigma_{j=1}^{v} r'_j a y_j d_j \neq 0$, so $\Sigma_{j=1}^{v} r'_j a y_j d_j \neq 0$. Hence, γ is indeed well-defined.

For all d in D, $(\gamma(\Sigma_j r'_j a y_j d_j))d = \Sigma_j r'_j b y_j d_j d = \gamma(\Sigma r'_j a y_j d_j d)$, so $\gamma \in R'$. But a similar argument shows that $[\gamma, R'] = 0$, implying $\gamma \in C$. Moreover, $(\gamma a - b)M = 0$; hence $\gamma a = b$.

(iv) Let us review briefly Martindale's definition of the central closure of R. Consider {left R-module homomorphisms from (2-sided) ideals of R to R}. Given such maps $f_1: B_1 \to R$ and $f_2: B_2 \to R$, we say $f_1 \sim f_2$ if, for some ideal $B \subseteq B_1 \cap B_2$, f_1 and f_2 have the same restriction to B. One can show that $\sim$ is a congruence with respect to addition and composition of functions, so we wind up with a ring structure $\hat{R}$ of equivalence classes (under $\sim$). Given r in R, let $f_r: R \to R$ be defined by $f_r(r') = r'r$, each r' in R. Then $r' \to [f_r]$ is an injection of R into $\hat{R}$. $\mathrm{Cent}(\hat{R})$ is the set of equivalence classes of bimodule homomorphisms from ideals of R to R, and is called the *extended centroid* of R; the subring of $\hat{R}$ generated by R and $\mathrm{Cent}\, \hat{R}$ is the *central closure* of R, and will be written $\tilde{R}$.

Now we define a map $\varphi: RC \to \tilde{R}$ in the most natural way possible. Suppose $\Sigma_{i=1}^{v} x_i \gamma_i \in RC$, nonzero x_i in R, nonzero γ_i in C. Let $B_i = R \cap R\gamma_i^{-1}$, a nonzero ideal of R (since R is essential subring of RC). Then $(\bigcap_i B_i)(\Sigma_{i=1}^{v} x_i\gamma_i) \subseteq R$, so $\Sigma_{i=1}^{v} x_i\gamma_i$ induces a left module homomorphism f: $\bigcap_i B_i \to R$, by right multiplication in RC. Let $\varphi(\Sigma_{i=1}^{v} x_i\gamma_i) = [f]$. Clearly, $\varphi: RC \to \hat{R}$ is a well-defined homomorphism. Suppose $u \in \ker\varphi$. Then $Bu = 0$ for some nonzero ideal B of R. But then $B(R \cap Ru) = 0$, implying $R \cap Ru = 0$; hence $u = 0$, so $\ker\varphi = 0$.

To show φ is onto, it suffices to prove the last assertion, that $\varphi(C) = \mathrm{Cent}\,\hat{R}$. Clearly $\varphi(C) \subseteq \mathrm{Cent}\,\hat{R}$. On the other hand, suppose $[f] \in \mathrm{Cent}\,\hat{R}$, i.e., f: $B \to R$ is a bimodule homomorphism, for a suitable ideal B of R. For any b in B, $f(b)rb = brf(b)$ for all r in B; hence, by (ii), $f(b) = \gamma b$, suitable γ in C. Moreover, γ is independent of the choice of b in B. Indeed, suppose $f(b_i) = \gamma_i$, for b_1, b_2 in B. For all r in B, $\gamma_1 b_1 r b_2 = f(b_1)rb_2 = b_1 r f(b_2) = \gamma_2 b_1 r b_2$, implying $(\gamma_1 - \gamma_2)b_1 B b_2 = 0$, so $\gamma_1 = \gamma_2$. Thus, there exists γ in C such that $f(b) = \gamma b$, all b in B. Clearly, $\varphi(\gamma) = [f]$, proving that $\varphi(C) = \mathrm{Cent}\,\hat{R}$, as desired. Q.E.D.

Theorem 10 shows that the central closure of any prime Johnson ring (in particular, any prime ring with a proper generalized identity) is canonically identified with the subring of T generated by R and Cent T, fulfilling our aim of tying the theory of the central closure into the theory of §1.

§2. Rings with pivotal monomial and related conditions

We recall definitions from [17]. Let $\mathbf{Z}\{X; t\}$ be the free ring generated by the noncommuting indeterminates $X_1, \cdots, X_t$; $\pi^k(t) = \{$monic monomials $h \in \mathbf{Z}\{X; k\} \mid h \neq X_1 \cdots X_t$ and $\deg h \geqq t\}$; R_1 is the ring with 1 formally adjoined to R. An element y in R_1 is *left R-regular* if $yr \neq 0$, all nonzero r in R; y is *strongly left R-regular* if $yr \neq 0$ and $ry \neq 0$, all nonzero r in R, and if Ry is large. Call $X_1 \cdots X_t$ *R-pivotal* (resp. *almost R-pivotal*) if, for each homomorphism φ: $\mathbf{Z}\{X; t\} \to R$, one can find strongly left R-regular (resp. left R-regular) $y \in R_1$, such that $y\varphi(X_1 \cdots X_t) \in R_1\varphi(\pi^t(t))$. If, for each φ, we can set $y = 1$, then $X_1 \cdots X_t$ is *absolutely R-pivotal*. If $X_1 \cdots X_t$ is R-pivotal for some t, we say *R satisfies a pivotal monomial.*

THEOREM [17, theor. 6]. *Let R be prime. $X_1 \cdots X_t$ is R-pivotal if and only if R is a left order in a simple artinian ring of index $\leqq t$ (where we define the* index *of a simple artinian ring $M_n(D)$ to be n).*

Applying this result to Theorem 5 yields

THEOREM 12. *The following conditions are equivalent for a prime ring R:*

(i) *R is Johnson;*

(ii) *R has a left ideal V such that $V/\mathrm{Ann}_V V$ satisfies a pivotal monomial;*

(iii) *R has a right ideal V such that* $V/\mathrm{Ann}'_V V$ *satisfies a pivotal monomial*;
(iv) *R has a principal left ideal V such that* $V/\mathrm{Ann}_V V$ *is an Ore domain*;
(v) *R has a principal right ideal V such that* $V/\mathrm{Ann}'_V V$ *is an Ore domain.*

Unfortunately, pivotal monomials do not pass to subrings (in particular to left and right ideals) in a natural way. Still, it is possible to obtain useful results by modifying the definitions. An element y of R_1 is *V-regular* if $yx \notin \mathrm{Ann}' V$ for all x in $V - \mathrm{Ann}' V$. Clearly, every left regular element is V-regular for all $V \subseteq R$.

PROPOSITION 13. *Let V be a right ideal of R. Suppose* $V/\mathrm{Ann}'_V V$ *is a prime ring and there exists* $k \geqq t$ *such that, for any homomorphism* $\varphi: \mathbf{Z}\{X; k\} \to V$, *we can find V-regular y in* R_1 *with* $y\varphi(X_1 \cdots X_t)V \subseteq R_1\varphi(\pi^k(t))V$. *Then every chain of left annihilators in* $V/\mathrm{Ann}'_V V$ *has length* $\leqq t+1$.

PROOF. (as in [17, prop. 3]). Let $\bar{\ }$ denote canonical images in $\bar{V} = V/\mathrm{Ann}'_V V$. Suppose that there exists a chain of left annihilators $\bar{V} \supset \bar{L}_1 \supset \bar{L}_2 \supset \cdots \supset \bar{L}_t \supset 0$ of length $t+2$, and let $\bar{T}_i = \mathrm{Ann}\,\bar{L}_i$, $1 \leqq i \leqq t$. (Note that $L_iT_iV = 0$, all i.) Pick arbitrarily $\bar{x}_i$ in $\bar{T}_i\bar{L}_i$, $1 \leqq i \leqq t$; set $x_i = 0$, $t < i \leqq k$. Define $\varphi: \mathbf{Z}\{X; k\} \to V$ by $\varphi(X_i) = x_i$, each i. Since $x_jx_i \in \mathrm{Ann}' V$ for each $j \geqq i$, we see that $yx_1 \cdots x_t \in \mathrm{Ann}' V$ for some V-regular y. Hence $x_1 \cdots x_t \in \mathrm{Ann}' V$, i.e., $\bar{x}_1 \cdots \bar{x}_t = 0$, so $\bar{T}_1(\bar{L}_1\bar{T}_2)\cdots(\bar{L}_{t-1}\bar{T}_t)\bar{L}_t = 0$, contrary to the fact that $\bar{V}$ is prime. Therefore, there is no chain of left annihilators of length $> t+1$. Q.E.D.

Let us refine further the idea of pivotal monomials. Given a right ideal V of R, call a V-regular element y of R_1 *Ore V-regular* if, for every subset B of R such that $VBV \neq 0$, we have r in R and b in R_1B, such that $VbV \neq 0$ and $(ry - b)V = 0$. If $V' \subseteq V$, then every V-regular element is clearly $V'V$-regular. Motivated by this fact, we call an Ore V-regular element y of R' *strongly Ore V-regular* if y is Ore $V'V$-regular for every subset $V' \subseteq V$.

Recall that a ring R is *Ore* when it satisfies the *Ore condition*: For any elements r_1, r_2, r_1 regular, one can find r_3, r_4 in R, r_3 regular, such that $r_4r_1 = r_3r_2$.

LEMMA 14. *Suppose V is a right ideal of R and* $\bar{V} = V/\mathrm{Ann}' V$ *is a semiprime Ore ring. Then, for any element y in V, if* $\bar{y}$ *is regular in* $\bar{V}$, *then y is strongly Ore V-regular.*

PROOF. Let $V' \subseteq V$. We need to prove that y is Ore $V'V$-regular. Well, let $B \subseteq R$ with $V'VBV'V \neq 0$. Then $v_1bv_2V \neq 0$ for suitable v_1 in V, b in B, v_2 in V'. By the Ore condition, there exist $\bar{v}_3$, $\bar{v}_4$ in $\bar{V}$, $\bar{v}_3$ regular, such that $\bar{v}_4\bar{y} = \bar{v}_3\overline{v_1b}$. Hence $(v_4y - v_3v_1b)V = 0$; in particular, $(v_4y - v_3v_1b)V'V = 0$. On the other hand, $\bar{v}_3\overline{v_1bv_2} \neq 0$, so $\overline{v_3v_1bv_2}\,\overline{Vv_3v_1bv_2} \neq 0$, implying $v_2Vv_3v_1bv_2V \neq 0$;

thus $V'Vv_3v_1bV'V \neq 0$. This proves y is Ore $V'V$-regular. It follows immediately that y is Ore V-regular; hence y is strongly Ore V-regular.

Q.E.D.

Define a right ideal V of R to be *pre-Goldie* (resp. *strongly pre-Goldie*) of degree t if there is $k \geqq t$, such that, given a homomorphism $\varphi: \mathbf{Z}\{X; k\} \to V$, we can find Ore V-regular (resp. strongly Ore V-regular) y in R_1 and r' in $R_1\varphi(\pi^k(t))$, with $(y\varphi(X_1 \cdots X_t) - r')V = 0$. Clearly, every right ideal of a ring with absolutely pivotal monomial is strongly pre-Goldie; also, each semiprime Goldie ring is itself strongly pre-Goldie, by [17, theor. 7] and Lemma 14.

Proposition 15. *If V is a pre-Goldie right ideal of R such that $\bar{V} = V/\mathrm{Ann}'_V V$ is prime, then $\bar{V}$ does not contain an infinite direct sum of nonzero left ideals.*

Proof. By Proposition 13, there is a maximal chain $0 = \bar{L}_0 \subset \cdots \subset \bar{L}_{n+1} = \bar{V}$ of left annihilators in $\bar{V}$. Since $\bar{L}_0$ contains no (nonzero) left ideals of $\bar{V}$, the proposition follows inductively from

Claim. *If $\bar{L}_{i+1}$ contains an infinite direct sum of left ideals of $\bar{V}$, then $\bar{L}_i$ contains an infinite direct sum of left ideals of $\bar{V}$.*

To prove the claim, let $L = \bar{L}_{i+1}$, and suppose $\bar{B} = \bigoplus_k \bar{B}_k \subseteq L$ is an infinite direct sum of left ideals of $\bar{V}$. Following the proof of [17, prop. 5], we search for a nonzero element of $\bar{B} \cap \bar{L}_i$. Let $\bar{T}_j = \mathrm{Ann}_L \bar{L}_j$, $0 \leqq j \leqq i+1$. For each u, $\bar{T}_i\bar{B}_u \not\subseteq \bar{T}_{i+1}$, so pick x_u in $\bar{T}_i\bar{B}_u - \bar{T}_{i+1}$. $L \supset \mathrm{Ann}'_L \bar{x}_u \supseteq \bar{L}_i$, implying $\mathrm{Ann}'_L \bar{x}_u = \bar{L}_i$, by [17, prop. 4(ii)].

Since V is pre-Goldie of degree t, we have some $k \geqq t$ such that, given $\varphi: \mathbf{Z}\{X; k\} \to V$, $(y\varphi(X_1 \cdots X_t) - r_t)V = 0$ for suitable Ore V-regular y in R_1 and r_t in $R_1\varphi(\pi^k(t))$. In particular, we define φ by $\varphi(X_u) = x_u$, $1 \leqq u \leqq t$, and $\varphi(X_u) = 0$, $t < u \leqq k$. Also, we can find m such that $\{\bar{x}_1, \cdots, \bar{x}_t\} \subseteq \bar{B}_1 \oplus \cdots \oplus \bar{B}_m$. $\bar{V}\bar{B}_{m+1} \neq 0$, implying $VR_1B_{m+1}V \neq 0$; by definition of Ore V-regular, we have b in R_1B_{m+1} and x in R, such that $(b - xy)V = 0$ and $VbV \neq 0$. Choose v_1, v_2 in V such that $v_1bv_2 \neq 0$. Now $\overline{v_1xy}\,\overline{x_1 \cdots x_t} \in \overline{v_1xR_1}\,\overline{\varphi(\pi^k(t))}$; comparing components of $\bar{B}$ yields $\overline{v_1xy}\,\overline{x_1 \cdots x_t} \in \overline{v_1xR_1}\,\overline{\varphi(\pi^k(t-1)}\bar{x}_t$, so $\overline{v_1xy}\,\bar{x}_1 \cdots \bar{x}_t = \overline{v_1xr_{t-1}}\bar{x}_t$, suitable r_{t-1} in $R_1\varphi(\pi^k(t-1))$. But then $(\overline{v_1xy}\,\overline{x_1 \cdots x_{t-1}} - \overline{v_1xr_{t-1}}) \in \bar{B} \cap \mathrm{Ann}'_L \bar{x}_t = \bar{B} \cap \bar{L}_i$, so our search is done unless $\overline{v_1xy}\,\overline{x_1 \cdots x_{t-1}} = \overline{v_1xr_{t-1}} \in \overline{v_1xR_1\varphi(\pi^k(t-1))}$. Continuing in this way, we have a nonzero element of $\bar{B} \cap \bar{L}_i$ unless $\overline{v_1xy}\,\bar{x}_1 \in \overline{v_1xR_1}\,\overline{\varphi(\pi^k(1))}$. Comparing components of $\bar{B}$ yields $\overline{v_1xy}\,\bar{x}_1 = \overline{v_1xr_0x_1}$, suitable r_0 in $R_1x_1 + \cdots + R_1x_t$. Since $\overline{v_1xy} - \overline{v_1xr_0} \in \bar{B} \cap \bar{L}_i$, our search is done unless $\overline{v_1xy} = \overline{v_1xr_0}$. But $\overline{v_1xy} = \overline{v_1b} \in \bar{B}_{m+1}$ and $\overline{v_1xr_0} \in \bar{V}\bar{x}_1 + \cdots + \bar{V}\bar{x}_t \subset$

$\bar{B}_1 \oplus \cdots \oplus \bar{B}_m$; hence $\overline{v_1 b} = 0$, contrary to $v_1 b v_2 \neq 0$. Thus we always have some nonzero element of $\bar{B} \cap \bar{L}_t$, which we call $\bar{b}_1$.

Now $\bar{b}_1 \in \bar{B}_1 \oplus \cdots \oplus \bar{B}_p$ for some p. The above argument shows that $(\bigoplus_{v>p} \bar{B}_v) \subseteq \bar{L}_t$ contains a nonzero element $\bar{b}_2$. Continuing in this way gives us an infinite number of $\bar{b}_j$ such that $\bigoplus_j \bar{V}\bar{b}_j \subseteq \bar{L}_t$. Thus, the claim is proved, so the proposition follows. Q.E.D.

(The careful reader may wish to note that the full force of the definition of pre-Goldie is not needed in the above proof.) We recall the *Faith–Utumi* Theorem:

Any (left) order in a simple artinian ring $M_n(D)$, D a division ring, contains a subring of the form $M_n(T)$, T an order in D.

THEOREM 16. *Let V be a right ideal of R, and assume $\bar{V} = V/\mathrm{Ann}'_V V$ is prime. The following conditions are then equivalent*:

(i) *V is pre-Goldie of degree $\leqq t$;*

(ii) *$\bar{V}$ is an order of a simple artinian ring of index $\leqq t$;*

(iii) *V is strongly pre-Goldie of degree $\leqq t$.*

PROOF. (i) $\Rightarrow$ (ii) By Propositions 13 and 15, and by Goldie's theorem, $\bar{V}$ is an order in a simple artinian ring; by Proposition 13 and the Faith–Utumi theorem, this ring has index $\leqq t$.

(ii) $\Rightarrow$ (iii) $X_1 \cdots X_t$ is $\bar{V}$-pivotal; by Lemma 14, this implies V is strongly pre-Goldie of degree $\leqq t$.

(iii) $\Rightarrow$ (i) Immediate. Q.E.D.

COROLLARY 17. *Every pre-Goldie right ideal of a prime ring is strongly pre-Goldie. A prime ring is Johnson if and only if it has a pre-Goldie right ideal.*

PROOF. Immediate consequence of Theorems 16 and 12.

COROLLARY 18. *If V_1, V_2 are right ideals of a prime ring, if V_2 is pre-Goldie, and if $V_1 \subseteq V_2$, then V_1 is pre-Goldie.*

PROOF. V_2 is strongly pre-Goldie, by Corollary 17, so $V_1 V_2$ is pre-Goldie; it follows easily that V_1 is pre-Goldie. Q.E.D.

One nice aspect of concepts like strongly pre-Goldie is that they provide a way of passing monomial conditions from rings to right ideals and their homomorphic images.

Meanwhile, let us see how pre-Goldie ideals of a Johnson ring fit into the endomorphism ring R' of §1.

LEMMA 19. *Suppose R is an essential subring of a ring T with* 1, *and R is prime with $Z(R)=0$. If V is a right ideal of R, then $V/\mathrm{Ann}'_V V$ can be embedded as an essential subring of the prime ring $VT/\mathrm{Ann}'(VT)$.*

PROOF. Let $\overline{VT}=VT/\mathrm{Ann}'(VT)$, and let $\bar{\ }$ denote the canonical image in $\overline{VT}$. Then $\bar{V}=(V+\mathrm{Ann}'(VT))/\mathrm{Ann}'\,VT\approx V/V\cap\mathrm{Ann}'\,VT= V/V\cap\mathrm{Ann}'\,V=V/\mathrm{Ann}'_V V$.

Moreover, if $0\neq\bar{x}\in\overline{VT}$, then $xv\neq 0$ for some v in V; $0\neq Jx\subseteq R$ for some large left ideal J of R, and $J(xv)\neq 0$ (since $Z(R)\neq 0$), implying $0\neq\overline{VJx}\subseteq\bar{V}$. Thus $\bar{V}$ is an essential subring of $\overline{VT}$. Q.E.D.

LEMMA 20. *If T is a primitive ring with nonzero socle and if V is a right ideal of T, then $V/\mathrm{Ann}'_V V$ is primitive with nonzero socle.*

PROOF. Standard.

THEOREM 21. *Suppose R is prime Johnson, an essential subring of a ring R' of endomorphisms of a left vector space over a division ring. If V is a pre-Goldie right ideal of R, of degree t, then $V/\mathrm{Ann}'_V V$ can be embedded as a left order in the simple artinian ring $VR'/\mathrm{Ann}'\,VR'$, which has index t, and VR' is a sum of t minimal right ideals. (In particular, $V\subseteq\mathrm{soc}(R')$.)*

PROOF. Let $\overline{VR'}=VR'/\mathrm{Ann}'VR'$ and $\bar{V}=(V+\mathrm{Ann}'VR')/VR'\approx V/\mathrm{Ann}'_V V$. By Lemma 19, $\bar{V}$ is an essential subring of $\overline{VR'}$, which is primitive with nonzero socle, by Lemma 20. But $\bar{V}$ contains no infinite direct sum of left ideals; hence, $\mathrm{soc}(\overline{VR'})$ is a sum of only a finite number of minimal left ideals of $\overline{VR'}$, implying $\overline{VR'}$ is simple artinian. Since every large left ideal of a prime Goldie ring contains a regular element, it follows that $\bar{V}$ is a left order in $\overline{VR'}$. Moreover, $\overline{VR'}$ has index $\leqq t$, in view of Theorem 16.

By an observation of Drazin [5], $X_1\cdots X_t$ is absolutely $\overline{VR'}$-pivotal. Hence, in view of [17, theor. 11], $(VR')^{t+1}\subseteq\mathrm{soc}\,R'$. But R' is a ring of endomorphisms of a vector space over a division ring, which implies $R'/\mathrm{soc}(R')$ is prime. Therefore $VR'\subseteq\mathrm{soc}(R')$, so VR' is a sum of minimal right ideals of R'. Since $X_1\cdots X_t$ is absolutely $\overline{VR'}$-pivotal, one sees easily that VR' is a sum of t minimal right ideals of R'. Q.E.D.

The converse of Theorem 21 is routine. Namely, if VR' is a sum of t minimal right ideals of R', then $\overline{VR'}$ is simple artinian of index $\leqq t$, so $\bar{V}$ is a prime Johnson ring, an essential subring of $\overline{VR'}$; a straightforward argument shows that $\bar{V}$ is a left order in $\overline{VR'}$ and is thus Goldie, so V is pre-Goldie of degree $\leqq t$.

§3. Structure of rings in terms of elementary sentences

In this section we will examine the structure of rings in terms of elementary sentences. In particular, we shall work towards a generalization of Martindale's theorem to describe when R is Johnson. Dealing with a ring R, our first-order language will include the operations $+$, $-$, and $\cdot$, as well as the relation $=$ (and of course the symbols $\wedge$, $\vee$, $\sim$, and the quantifiers $\forall$, $\exists$). There are two obvious choices for the constant symbols. Certainly we want 0 and 1 (and thus each $n \in \mathbf{Z}$). In this case the study of universal atomic sentences is merely the theory of polynomial identities of R; the study of atomic $\exists\forall$ sentences is the theory of generalized polynomial identities. If we expand the set of constant symbols to all of R, then the universal atomic sentences are the generalized polynomial identities; "$\exists\forall$" does not add new sentences because the symbol $\exists$ at the beginning has the effect of adding new constants. The somewhat more intricate study of "pivotal monomials" is involved with $\exists\forall$ sentences with constant symbols in $\mathbf{Z}$; this paper is concerned with $\forall\exists$ sentences with constant symbols in R (or, equivalently, $\exists\forall\exists$ sentences with constant symbols in $\mathbf{Z}$), which include the "generalized pivotal monomials."

The following synopsis gives an idea of the direction of the structural results on a ring R. There are always conditions on the sentences being "proper" or "nontrivial", but we do not state them in the summary.

I. Polynomial identities ($\forall$ sentences, constant symbols in $\mathbf{Z}$)

A. Primitive implies central simple (Kaplansky's theorem)

B. Prime implies the "central localization" is central simple.

II. Generalized polynomial identities ($\forall$ sentences, constants in R)

A. Primitive implies R is dense in $\operatorname{End} M_D$, where M is a vector space over a central simple division algebra D, and $\operatorname{soc} R \neq 0$. (Amitsur's theorem [2].)

B. Prime implies Martindale's "central closure" of R is primitive, with proper generalized polynomial identity (as above). (Martindale's theorem [15].)

III. Pivotal monomials ($\forall\exists$ sentences, constants in $\mathbf{Z}$)

A. Primitive implies simple artinian [5].

B. Prime implies Goldie [17]. (However, the situation concerning pivotal monomials of prime Goldie rings is quite intricate, cf. [17].)

IV. Pivotal generalized monomials ($\forall\exists$ sentences, constants in R)

A. Primitive implies nonzero socle ([2], [17]).

B. Prime rings are to be discussed in this section.

Before continuing, one should remark that pivotal monomials do not comprise all atomic $\forall\exists$ sentences, many of which do not behave well in the above structural context. Indeed, an example is the von Neumann regular ring, defined by $(\forall x \exists y)(xyx = x)$. An interesting example of a primitive von Neumann regular ring with zero socle is given in [4], and it is unknown whether or not there exist prime von Neumann regular rings which are not primitive.

Also, there are results in the above theory, concerning "evaluations" of "generalized pivotal monomials". (In primitive rings, they are all in the socle; see [16], [17] for details.) These terms will be defined shortly, and will be studied in this section.

Clearly IVB in the above outline provides the link between sections 1 and 2 of this paper and the study of rings via elementary sentences. Already, some of the results obtained earlier can be translated into $\forall\exists$ sentences (with coefficients in R). Note that an element y in R is Ore aR-regular (for suitable a in R) iff

$$(\forall b, r \in R)(\exists r_1, r_2 \in R)((arba \neq 0)$$

$$\rightarrow (ar_1ba \neq 0 \wedge (r_2y - r_1b)a = 0)).$$

Thus, "Ore aR-regular" is elementary, and, using [17, theor. 6], theorem 16, and corollary 18, we have readily

THEOREM 22. *A prime ring R is Johnson iff, for suitable $a \neq 0$,*

$$(\forall x_1, \cdots, x_t)(\exists r_1, \cdots, r_u, y)((y \text{ is Ore } aR\text{-regular})$$

$$\wedge ((yax_1 \cdots ax_t - \sum_i r_i ax_{i_1} \cdots ax_{i_m})a = 0))$$

is satisfied for suitable t and u, with m varying between t and some suitable fixed number t', the sum taken over all m-tuples $(i_1, \cdots, i_m) \neq (1, 2, \cdots, t)$.

This sentence is not atomic, although we could make it atomic by using 1 in place of y; this would yield a sufficient (but not necessary) condition for a prime ring to be Johnson. A sentence of this type is known as a "generalized pivotal monomial", defined and studied in [17, sec. 4] as well as in related work by Amitsur [2], Desmarais–Martindale [4], and Jain. Thus, a natural question to ask is, "Does the presence of a generalized pivotal monomial imply that a prime ring is Johnson?"

Our analysis is based on [17, sec. 4], with appropriate modifications (to deal with prime rings in place of primitive rings). Given a ring R without 1, let R_1 be the ring formed by adjoining 1 formally, and let $R_1\{X\}$ denote the free product of R_1 and the free (noncommutative) ring $\mathbf{Z}\{X\}$. Each element f of $R_1\{X\}$ is a sum of "monomials" of the form $r_1X_{i_1}r_2X_{i_2}\cdots r_tX_{i_t}r_{t+1}$, for suitable r_i in R_1; call the r_i the *coefficients* and call $X_{i_1}\cdots X_{i_t}$ the *fingerprint* of the monomial. A *generalized monomial* is a sum of monomials with the same fingerprint. Any finite set W of R_1 containing the coefficients of h is called a *coefficient set* of h. In general, we will be interested in a generalized monomial coupled with a specific coefficient set.

Let $\pi(t, R_1; W) = \{$generalized monomials h of $R_1\{X\}$ with coefficient set W: the fingerprint of h has the form $X_{\pi 1} \cdots X_{\pi t}$, π a nonidentity permutation of $(1, \cdots, t)\}$. A generalized polynomial h in $R_1\{X\}$ having fingerprint $X_1 \cdots X_t$, with a coefficient set W, is R-*pivotal* if, for each homomorphism $\varphi: R_1\{X\} \rightarrow R_1$ sending each X_i to an element of R, we have $\varphi(h) \in R_1\varphi(\pi(t, R_1; W))$. Call $\varphi(h)$ an *evaluation* of h; h is R-*proper* if h has a nonzero evaluation (in R).

From now on, every generalized monomial under consideration has fingerprint $X_1X_2 \cdots$. If h can be written as a sum of v monomials with coefficients in W, with v minimal, define $ht_W(h) = v(t+1)$. Usually W will be understood, and we will merely write $ht(h)$.

Through Theorem 25, *we will assume that R is a prime ring which is an essential subring of a primitive ring P.* Let M be a faithful, irreducible P-module and let $D = \mathrm{End}_P M$, a division ring. Let P' be the subring of $\mathrm{End}\, M_D$ generated by P and 1; viewing P' and D canonically in $\mathrm{End}_{\mathbf{Z}} M$, let $S = P'D$. Since $pd = dp$ for all p in P' and d in D, S is a ring. Also, M is an S-module under the action $(\Sigma_i p_i d_i)z = \Sigma_i (p_i z) d_i$ for all p in P', d_i in D, z in M.

Let $\mathcal{F} = \{$large left ideals of $R\}$.

LEMMA 23. *For any A in $\mathcal{F}$, any finite dimensional D-subspace V of M, if $pAM \subseteq V$ then $pM \subseteq V$.*

PROOF. For any p_1 in P such that $p_1 V = 0$, we have $p_1 pAM = 0$, so $p_1 pA = 0$; since R is prime, it follows easily that $p_1 p = 0$, so $p_1(pM) = 0$. Therefore, by the density theorem, $pM \subseteq V$. Q.E.D.

We shall use Lemma 23 without further mention. Note that we can view $W \subseteq R \subseteq P \subseteq P' \subseteq S \subseteq S\{X\}$, and have a map $R_1 \rightarrow P'$ given by $(n, r) \mapsto n + r$; then there are induced homomorphisms (not necessarily injective) from $R_1\{X\}$, $P'\{X\}$, and D, to $S\{X\}$. Under these maps, call $P'\{X\}D$ the additive subgroup of $S\{X\}$ generated by elements fd, for all f in $P'\{X\}$, d in D. Given a generalized monomial h of $P\{X\}D$ and a coefficient set W of h, let U be the D-subspace of $P'\{X\}D$ generated by W and 1. We say a generalized monomial g *matches* h if g is another way of writing h (in $P'\{X\}D$) with coefficients in U. (W need not be a coefficient set of g.) Define $c(h) = \dim U_D$. This will replace "n" of [17, theor. 10] and, in fact, should have been used in that proof.

DEFINITION 24. A generalized monomial $h(X_1, \cdots, X_t)$ with coefficient set W is $(V, (u_i))$-*dominated* for natural numbers $u_1, \cdots, u_t$, if there exist D-subspaces V_i (of M) having respective dimension u_i, and $A \in \mathcal{F}$ with the following property: For each homomorphism $\varphi: S\{X\} \to S$ with $\varphi(X_i) \in R$ and $\varphi(X_i)V_i = 0$, $1 \leqq i \leqq t$, and for every a in A and z in M, we can find r in $R'\varphi(\pi(t, S; W))$ such that $(\varphi(h) - r)az \in V$.

NOTE. If two generalized monomials h_1 and h_2, with coefficient set W, are $(V, (u_i))$-dominated, then $h_1 + h_2$ is also $(V, (u_i))$-dominated.

THEOREM 25. *Suppose V is a subspace (of M) of dimension u_0 and h is a generalized monomial in $P'\{X\}D$ with coefficient set W. Also suppose h has fingerprint $X_1 \cdots X_t$ and is $(V, (u_i))$-dominated. Let $b = ht(h)$, $c = c(h)$, and $u = \max\{u_0, u_1, \cdots, u_t\}$. There is a function $\tau_{u,c}: \mathbf{Z}^+ \to \mathbf{Z}^+$ and a generalized monomial g matching h, such that every monomial of g has a coefficient of rank $\leqq \tau_{u,c}(b)$.*

PROOF. (Based on [17, theor. 10].) Let h be a sum of v monomials, and let $V_1, \cdots, V_t$ be as in Definition 24. Define $\tau_{u,c}(0) = 0$ for all u and c, and define inductively

$$\tau_{u,c}(b) = \max(ub, \tau_{u+c,c}(b-1), \tau_{u',c}(b') \quad \text{for all } u' < u \text{ and all } b' \leqq b).$$

If $b = 1$ then the theorem is immediate. We work by induction on b.

Write $h = \Sigma_{j=1}^{n} h_j(X_1, \cdots, X_{t-1})X_t s_j$ such that each h_j has coefficient set W and, if h_j is a sum of v_j monomials, $\Sigma v_j = v$. We may assume, for some $m \leqq v$, that for all j with $m < j \leqq v$, $s_j M \subseteq V_t$; m is chosen minimal in this context.

Let $h' = \Sigma_{j=1}^{m} h_j X_t s_j$ and $h'' = h - h'$. Clearly h'' is $(0, (u_i))$-dominated, so h' is $(V, (u_i))$-dominated and $ht(h') + ht(h'') = b$. If $h' \neq 0$ and $h'' \neq 0$ then, by induction, h' and h'' are each matched to generalized monomials g' and g'', all of whose monomials have a coefficient of rank $\leqq \max(\tau_{u,c}(ht\, b'), \tau_{u,c}(ht\, b'')) \leqq \tau_{u,c}(b)$, so we are done with $g = g' + g''$; if $h' = 0$ then we are done by setting $g = h'' = h$. Hence we may assume $h = h'$.

Let A be as in Definition 24, and choose z_0 arbitrarily in M. Since $s_1 AM \not\subseteq V_t$, we have $s_1 az \notin V$ for some a in A, some z in M. By density, there exist p_t in P, $d_1, \cdots, d_n$ in D, with $d_1 = 1$, such that $p_t V_t = 0$ and $p_t s_k az = z_0 d_k$ for all k, $1 \leqq k \leqq n$. Let $A_1 = Rp_t^{-1} \in \mathcal{F}$. Also let

$$h_1' = \Sigma_{j=1}^{n} h_j d_j, \qquad V_t' = V_t + \Sigma_{k=1}^{n} s_k azD,$$

$$\text{and } u_t' = \dim V_t' \leqq u_t + c.$$

Clearly h_1' has coefficients in $W' = \{wd_j \mid w \in W, 1 \leqq d_j \leqq n\}$, so we replace W by W', enabling us to write $h_1' X_t s_1$ as a sum of monomials with coefficients in W.

In order to conclude the proof exactly the same way as in [17, theor. 10] (writing $h = h_1'X_ts_1 + (h - h_1'X_ts_1)$ and proving that the conclusion of the theorem holds for each part), it suffices to show that h_1' is $(V,(u_t'))$-dominated.

Well, suppose $\varphi: S\{X\} \to S$ is a homomorphism sending X_t to an element x_t in R such that $x_tV_t' = 0$, $1 \leqq i \leqq t-1$. Given a_1 in A_1, since z_0 is arbitrary, it is enough to find r_0 in $R'\varphi(\pi(t-1, S; W))$ such that $(\varphi(h_1') - r_0)a_1z_0 \in V$. Since X_t has not yet appeared in this consideration, we may assume $\varphi(X_t) = a_1p_t$. Now, by Definition 24, we have r in $R'\varphi(\pi(t, S; W))$, with $(\varphi(h) - r)az \in V$; clearly r has the form $\Sigma r_k(a_1p_t)s_k$, where $r_k \in R'\varphi(\pi(t-1, S; W))$ and $s_k \in W$. By choice of the d_k, $raz = \Sigma r_ka_1p_ts_kaz = \Sigma r_ka_1z_0d_k = \Sigma r_kd_k(a_1z_0)$. Let $r_0 = \Sigma r_kd_k$. Then $raz = r_0a_1z_0$, so

$$
\begin{aligned}
&(h_1'(x_1,\cdots,x_{t-1}) - r_0)a_1z_0\\
&\quad = (\Sigma h_j(x_1,\cdots,x_{t-1})(a_1'z_0)d_j) - r_0a_1z_0\\
&\quad = \Sigma h_j(x_1,\cdots,x_{t-1})a_1p_ts_jaz - raz\\
&\quad = (\varphi(h) - r)az \in V,
\end{aligned}
$$

as desired. Q.E.D.

The major application of Theorem 25 is with $V = 0$ and all $u_t = 0$.

Corollary 26. *Suppose R is a prime ring, and R is an essential subring of a primitive ring P. Then every evaluation of every pivotal generalized monomial of R lies in $R \cap \operatorname{soc} P$. In particular, if R has a proper pivotal generalized monomial, then R is (prime) Johnson.*

(In many applications, the pivotal generalized monomial will be merely a monomial. The common terminology in this case is "generalized pivotal monomial.")

Call a prime ring "nice" if it is an essential subring of a primitive ring. Theorem 25 and Corollary 26 lead us to the following question:

Question. Are all prime rings nice?

An affirmative answer to this question would complete the theory of generalized pivotal monomials on prime rings. Goodearl (*Prime ideals in regular self-injective rings*, Canad. J. Math. **25** (1973), 829–839) proved that if R is prime and $Z(R) = 0$, then the Johnson–Utumi ring of quotients of R is primitive. In particular, every prime ring with singular ideal 0 is nice, and we can apply the above theory.

Another way of stating Goodearl's theorem is, "If $Z(R) = 0$ and R is prime, then R is an essential subring of a primitive von Neumann regular ring." Unfortunately, the converse is also true. Namely, if R is an essential subring of a von Neumann regular ring, then $Z(R)=0$. (Indeed, it is easy to see that the singular ideal of every von Neumann regular ring is 0, and that if R is an essential subring of S then $Z(R)\subseteq Z(S)$.) This fact complicates an attempt to settle the above question.

Let $B(R)=\{$evaluations of pivotal generalized monomials in $R\}$. Clearly $B(R)$ is a right ideal of R, and $B(R)\neq 0$ if and only if R has proper pivotal generalized monomials. If $1\in B(R)$, then there is [17, theor. 13] and the following parallel theorem:

THEOREM 27. *If R is a subdirect product of nice prime rings $\{R_\gamma \mid \gamma\in\Gamma\}$ and if $1\in B(R)$, then $\{R_\gamma \mid \gamma\in\Gamma\}$ are left Goldie, of bounded index.*

PROOF. Each R_γ is an essential subring of a primitive ring P_γ. The proof concludes exactly as in [17, theor. 13], to show that $\{P_\gamma \mid \gamma\in\Gamma\}$ are simple artinian of bounded index, so $\{R_\gamma \mid \gamma\in\Gamma\}$ are left Goldie of bounded index.

Q.E.D.

Having obtained results on rings with $B(R)\neq 0$ or with $1\in B(R)$, now we shall look for examples of such rings, thereby setting the stage for applications. Unfortunately, the well-known example following [17, lemma 14] is a counterexample to the converse of Theorem 27 and is also a counterexample to the converse of Corollary 26. A slightly different path is more inviting:

PROPOSITION 28. *X_1X_2 is a generalized pivotal monomial of every semisimple artinian ring.*

PROOF. Let $M_n(D)$ denote the ring of $n\times n$ matrices over a division ring D, with matric units $\{e_{ij} \mid 1\leqq i,j\leqq n\}$. Clearly it suffices to prove that X_1X_2 is pivotal (as a generalized monomial) for $M_n(D)$. Indeed, for any x_1, x_2 in $M_n(D)$, we have $x_1x_2=\sum_{i,j=1}^n d_{ij}e_{ij}$ for suitable d_{ij} in D. Hence, for suitable elements d_{ijkq} in D, we have

$$x_1x_2=\sum_{i,j,k,q=1}^{n} d_{ijkq}e_{ii}x_2e_{jk}x_1e_{qq},$$

proving the assertion. Q.E.D.

THEOREM 29. *Suppose there exists an element r in R and a polynomial $g(\lambda)$ in* $(\mathrm{Cent}(R))[\lambda]$ *such that $g(r)=0$, $g'(r)$ is right invertible (where g' is the formal derivative of g), and the centralizer of r in R is semisimple artinian. Then $1\in B(R)$. (In particular, each nice prime image of R is Goldie, and every primitive image of R is simple artinian.)*

PROOF. (mimicking [16, theor. 4]). Let S be the centralizer of r in R. By Proposition 28, X_1X_2 is generalized S-pivotal. Let $g(\lambda)=\sum_{i=0}^{n} c_i\lambda^i$, c_i in Cent(R), and define $f(X)=\sum_{k=1}^{n} c_k \sum_{i=0}^{k-1} r^iXr^{k-1-i}$. For any element x in R, $f(x)\in S$, so it follows that $f(X_1)f(X_2)$ is a pivotal generalized monomial of R. But $f(1)f(1)=(g'(r))^2$, an invertible element of R, implying $1\in B(R)$. The rest of the theorem is Theorem 27 and [17, theor. 13]. Q.E.D.

NOTE. Using very different methods, Miriam Cohen proved that if, under the hypotheses of Theorem 29, we also assume that R is semiprime and is an algebra over a field, then R is semisimple artinian. (In particular, every prime image of R is simple artinian.)

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